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On Schur flows

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Abstract. For the finite Schur (dmKdV) flows, a non-local Poisson structure is introduced and shown to be linked via Bäcklund–Darboux transformations to linear and quadratic Poisson structures for the Toda lattice. Two different Lax representations for the Schur flows are used, one to construct Bäcklund–Darboux transformations and the other to solve the Cauchy problem via the trigonometric moment problem.

1. Introduction

In this paper we deal with the system of nonlinear differential-difference equations

$$\dot{\gamma}_n = (1-\gamma_n^2)(\gamma_{n+1}-\gamma_{n-1}) \tag{1.1}$$

known as the *discrete modified KdV equation* (see, e.g., [1, 6, 7, 10]) or the equation of *the Schur flows* (cf [2, 3]). We will use the latter name because of the connection between (1.1) and the Schur parameters for the trigonometric moment problem.

Equations (1.1) first appeared in [1]. They belong to the family of integrable nonlinear differential-difference equations, whose most famous representatives are the Toda lattice equations

$$\dot{a}_n = a_n(b_{n+1}-b_n) \qquad \dot{b}_n = a_n - a_{n-1} \tag{1.2}$$

and the Volterra (or Kac–van Moerbecke) lattice

$$\dot{\alpha}_n = \alpha_n(\alpha_{n+1}-\alpha_{n-1}). \tag{1.3}$$

The construction and study of the Poisson geometry of Bäcklund–Darboux transformations between integrable lattice equations has attracted a lot of attention lately (see, e.g., [8, 11, 13]). Our aim is to introduce a non-local Poisson structure for the Schur flows (1.1), that can be linked to both linear and quadratic compatible Poisson brackets for the Toda lattice.

This structure is defined in section 2, where we also discuss relations between two Lax representations for the Schur flows. One of these representations, where the Lax matrix is Hessenberg, allows one to solve the initial value problem for finite or semi-infinite Schur flows using the trigonometric moment problem. This approach, similar to the treatment of the Toda lattice and the moment problem on the line (cf [4, 5]), is described in section 3.

The second Lax representation for the Schur flows, which involves a tridiagonal Lax operator, is more suitable for construction of Darboux-type transformations that map solutions of the Schur flows into solutions of the Volterra lattice. Properties of these transformations with respect to the non-local Poisson structure for the Schur flows are studied in section 4.

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2. Lax formulation and Poisson structures

We will concentrate on the finite non-periodic Schur flows

$$\dot{\gamma}_n = (1-\gamma_n^2)(\gamma_{n+1}-\gamma_{n-1}) \qquad 0 \leqslant n < N \qquad \gamma_{-1} = 1 \qquad \gamma_N = \epsilon = \pm 1. \tag{2.1}$$

Some of the statements below also remain valid for the semi-infinite Schur flows ($N = \infty$). The second boundary condition can be omitted in this case.

System (1.1) admits several Lax pair representations. We present two of them. In the first one, suggested in [10], the Lax operator is a tri-diagonal matrix $L = (l_{ij})_{i,j=0}^N$ of a special form:

$$l_{ij} = (1-\gamma_i^2)\delta_{i+1}^j + (\gamma_{i-1}-\gamma_i)\delta_i^j + \delta_{i-1}^j \tag{2.2}$$

and (1.1) is equivalent to the Lax equation

$$\dot{L} = [L, \pi_+(L^2)] \tag{2.3}$$

where $\pi_+(X)$ denotes the strictly upper triangular part of the matrix X.

Lax operator in the second representation is an upper Hessenberg matrix $U = (u_{ij})_{i,j=0}^N$:

$$u_{ij} = \begin{cases} -\gamma_{i-1}\gamma_j \prod_{k=i}^{j-1}(1-\gamma_k^2) & \text{for} \quad i \leqslant j \\ 1 & \text{for} \quad i = j+1 \\ 0 & \text{for} \quad i > j+1 \end{cases} \tag{2.4}$$

and the Lax equation reads

$$\dot{U} = [\pi_+(U + U^{-1}), U]. \tag{2.5}$$

If

$$1-\gamma_i^2 > 0 \qquad \text{for} \quad 0 \leqslant i < N \tag{2.6}$$

then U is similar to an orthogonal matrix $O = DUD^{-1}$, where $D = \mathrm{diag}(1, (1-\gamma_0^2)^{1/2}, (1-\gamma_0^2)^{1/2}(1-\gamma_1^2)^{1/2}, \ldots)$. O was proposed as a Lax matrix for the Schur flows in [2, 3].

The next proposition describes the relation between Lax operators (2.3) and (2.5).

Proposition 2.1. *There exists a matrix X with entries depending polynomially on variables* γ_i *such that*

$$X(U+U^{-1})X^{-1} = L^2 - 2.$$

Proof. Define matrices

$$\Gamma_i = \begin{pmatrix} -\gamma_i & 1-\gamma_i^2 \\ 1 & \gamma_i \end{pmatrix} \qquad (i = 0, \ldots, N-1)$$

$$P_i = \mathrm{diag}(\mathbf{1}_i, \Gamma_i, \mathbf{1}_{N-i-1}) \qquad (i = 0, \ldots, N-1)$$

and

$$P_n = \mathrm{diag}(\mathbf{1}_N, -\gamma_N).$$

Then

$$P_i^2 = \mathbf{1} \quad (i = 0, \ldots, N) \qquad [P_i, P_j] = 0 \quad (|i-j| > 1). \tag{2.7}$$

Moreover, $U = P_0 \cdots P_N$ and $U^{-1} = P_N \cdots P_0$. Put

$$X = \overleftarrow{\prod}_{k=1}^{[N/2]} (P_{N-2k} P_{N-2k-1} \cdots P_0).$$

Then relations (2.7) imply

$$\tilde{U} = X(U + U^{-1})X^{-1} = \Pi_1\Pi_2 + \Pi_2\Pi_1$$

where $\Pi_1 = \prod_{k=1}^{[N/2]} P_{N-2k-1}$, $\Pi_2 = \prod_{k=1}^{[N+1/2]} P_{N-2k}$. A direct computation shows that $\tilde{U} = L^2 - 2$. □

In what follows, we fix the second boundary condition in (2.1):

$$\epsilon = 1. \tag{2.8}$$

Proposition 2.2. *Local Poisson structure for (2.1) with $\epsilon = 1$ can be defined as follows:*

$$\{\gamma_i, \gamma_{i+1}\}_l = -(1 - \gamma_i^2)(1 - \gamma_{i+1}^2) \tag{2.9}$$

and all other Poisson brackets are zero. Equations of the Schur flow are generated by the Hamiltonian function

$$H_l = \sum_{i=0}^{N-1} \ln(1 + \gamma_i). \tag{2.10}$$

Proof. By direct computation. □

We now define a non-local skew-symmetric bracket for γ_i:

$$\{\gamma_i, \gamma_j\}_{nl} = -(1 - \gamma_i)(1 - \gamma_j) \prod_{k=i+1}^{j-1} \frac{1 - \gamma_k}{1 + \gamma_k} \qquad (i < j). \tag{2.11}$$

Proposition 2.3. *Equation (2.11) defines a Poisson bracket. Equations (1.1) are Hamiltonian with respect to this bracket with the Hamilton function*

$$H = -\operatorname{Tr}(U) = 2(N + 1) - \operatorname{Tr}(L^2) = \sum_{i=0}^{N} \gamma_i \gamma_{i-1}. \tag{2.12}$$

Proof. First note, that for any pair of functions f, g a skew-symmetric bracket of the form

$$\{\gamma_i, \gamma_j\}_{nl} = -f(\gamma_i) f(\gamma_j) \prod_{k=i+1}^{j-1} g(\gamma_k) \qquad (i < j)$$

extended via the Leibnitz rule satisfies the Jacobi identity and, thus, defines a Poisson bracket. This can checked through a straightforward though tedious computation.

Next, denote

$$\pi_{ij} = \prod_{k=i}^{j} \frac{1 - \gamma_k}{1 + \gamma_k}. \tag{2.13}$$

Then

$$\{\gamma_i, H\}_{nl} = \sum_{j=0}^{N-1} \{\gamma_i, \gamma_j\}_{nl} (\gamma_{j-1} + \gamma_{j+1})$$
$$= (1-\gamma_i)\Bigg(\sum_{j=0}^{i-1} (1-\gamma_j)(\gamma_{j-1}+\gamma_{j+1})\pi_{j+1,i-1}$$
$$- \sum_{j=i+1}^{N-1} (1-\gamma_j)(\gamma_{j-1}+\gamma_{j+1})\pi_{i+1,j-1}\Bigg).$$

Note, that

$$(1-\gamma_j)(\gamma_{j-1}+\gamma_{j+1})\pi_{j+1,i-1} = (1-\gamma_j)(1-\gamma_{j+1})\pi_{j+2,i-1} - (1-\gamma_{j-1})(1-\gamma_j)\pi_{j+1,i-1}$$
$$(j < i-1)$$

and

$$(1-\gamma_j)(\gamma_{j-1}+\gamma_{j+1})\pi_{i+1,j-1} = -(1-\gamma_j)(1-\gamma_{j+1})\pi_{i+1,j-1} + (1-\gamma_{j-1})(1-\gamma_j)\pi_{i+1,j-2}$$
$$(i < j-1).$$

Thus

$$\dot{\gamma}_i = \{\gamma_i, H\}_{nl} = (1-\gamma_i)(-(1-\gamma_{-1})(1-\gamma_0)\pi_{1,i-1}$$
$$+(1-\gamma_{i-2})(1-\gamma_{i-1}) + (1-\gamma_{i-1})(\gamma_{i-2}+\gamma_i) - (1-\gamma_{i+1})(\gamma_i+\gamma_{i+2})$$
$$+(1-\gamma_{i+1})(1-\gamma_{i+2}) + (1-\gamma_{N-1})(1-\gamma_N)\pi_{i+1,N-2})).$$

Taking into account boundary conditions (2.1) and (2.8), we obtain equations (1.1). □

3. Schur flows and trigonometric moment problem

First, we review some basic facts about the trigonometric moment problem. Details can be found, for example, in [9, 12]. Consider a sequence of real numbers τ_n, $n \geqslant 0$, put

$$\tau_{-n} = \tau_n \tag{3.1}$$

and define Toeplitz matrices

$$T_n = (\tau_{i-j})_{i,j=0}^{n}. \tag{3.2}$$

We assume that

$$\Delta_n = \det T_n \neq 0 \tag{3.3}$$

for $0 \leqslant n < N+1$.

One can define a bilinear functional $\langle\, ,\, \rangle$ on the set $\mathbb{R}[z, z^{-1}]$ of Laurent polynomials by putting

$$\langle z^i, z^j \rangle = \tau_{i-j}$$

and then extending by linearity. Note that the functional thus defined is invariant under multiplication by z:

$$\langle zp(z), zq(z)\rangle = \langle p(z), q(z)\rangle \qquad \text{for any} \quad p(z), q(z) \in \mathbb{R}[z, z^{-1}]. \tag{3.4}$$

Then monic polynomials defined by the formulae

$$\phi_n(z) = \frac{1}{\Delta_{n-1}} \det \begin{bmatrix} \tau_0 & \tau_1 & \cdots & \tau_n \\ \tau_1 & \tau_0 & \cdots & \tau_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \tau_{n-1} & \tau_{n-2} & \cdots & \tau_1 \\ 1 & z & \cdots & z^n \end{bmatrix} \tag{3.5}$$

satisfy the following properties:

(a) ϕ_n are orthonormal with respect to $\langle\ ,\ \rangle$:

$$\langle \phi_i, \phi_j \rangle = 0 \qquad j \neq i.$$

(b) If we introduce *Schur parameters* $\nu_i = -\phi_i(0)$, then

$$\langle \phi_{i+1}, \phi_{i+1} \rangle = (1 - \nu_{i+1}^2) \langle \phi_i, \phi_i \rangle \qquad i = 0, 1, \ldots. \tag{3.6}$$

(c) For a polynomial $p(z)$ of degree n define $p^\sharp(z) = z^n p(1/z)$. Then polynomials $\phi_n, \phi_n^\sharp$ satisfy the recurrence relations:

$$\phi_n = z\phi_{n-1} - \nu_n \phi_{n-1}^\sharp \tag{3.7}$$

$$\phi_n^\sharp = z\phi_{n-1}^\sharp - z\nu_n \phi_{n-1} \tag{3.8}$$

and

$$\phi_n^\sharp = -\sum_{i=0}^{n} \nu_i \frac{\langle \phi_n, \phi_n \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$z\phi_n = \phi_{n+1} - \nu_{n+1} \sum_{i=0}^{n} \nu_i \frac{\langle \phi_n, \phi_n \rangle}{\langle \phi_i, \phi_i \rangle} \phi_i. \tag{3.10}$$

Thus, if one denotes

$$\nu_i = -\gamma_{i-1}$$

then it follows from (3.6) and (3.10) that matrix elements of the operator of multiplication by z written in the basis $\phi_0, \phi_1, \ldots$ coincide with u_{ij} defined by (2.4). Moreover, equation (3.10) together with (3.5) imply

$$\gamma_n = \frac{\langle z\phi_n, 1 \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{(-1)^{n+1}}{\Delta_n} \det \begin{bmatrix} \tau_1 & \tau_2 & \cdots & \tau_{n+1} \\ \tau_0 & \tau_1 & \cdots & \tau_n \\ \cdots & \cdots & \cdots & \cdots \\ \tau_{n-2} & \tau_{n-3} & \cdots & \tau_2 \\ \tau_{n-1} & \tau_{n-2} & \cdots & \tau_1 \end{bmatrix} \tag{3.11}$$

and

$$\tau_n = \langle z^n, 1 \rangle = (U^n)_{00}. \tag{3.12}$$

In the case when the sequence (τ_i) is positive definite, i.e. all determinants in (3.3) are positive, there exists a non-negative measure $\mathrm{d}\rho(z)$ on the circle $S^1 = \{|z| = 1\}$ such that

$$\tau_n = \int_{S^1} z^n \, \mathrm{d}\rho(z). \tag{3.13}$$

Then condition (3.1) is equivalent to a symmetry condition for the measure $\mathrm{d}\rho(z)$:

$$\mathrm{d}\rho(z) = \mathrm{d}\rho(\bar{z}). \tag{3.14}$$

In the case, when N is finite, a support of $\mathrm{d}\rho(z)$ consists of $N+1$ points of the unit circle symmetric under the complex conjugation and weights associated with conjugated points of the support are equal.

Formulae (3.11) and (3.12) allow one to integrate equations (1.1) of the Schur flows. Indeed, if U satisfies the Lax equation (2.5), then the same is true for U^n. Thus,

$$\begin{aligned}\dot{\tau}_n &= ([\pi_+(U+U^{-1}), U^n])_{00} = (\pi_+(U+U^{-1})U^n)_{00}\\ &= (U^{n+1}+U^{n-1})_{00} - (\pi_-(U+U^{-1})U^n)_{00}.\end{aligned}$$

Here $\pi_-(X)$ is a projection on the lower triangular part of X. Observing that $(\pi_-(U + U^{-1})U^n)_{00} = (U+U^{-1})_{00}(U^n)_{00} = -2\gamma_0\tau_n = 2\tau_1\tau_n$, we obtain

$$\dot{\tau}_n = \tau_{n+1} + \tau_{n-1} - 2\tau_1\tau_n. \tag{3.15}$$

For an alternative derivation see [3].

Define

$$\Gamma(t) = \exp\big(t(U(0) + U^{-1}(0))\big). \tag{3.16}$$

It is not hard to check that the solution to (3.15) with initial data $\tau_n(0) = (U^n(0))_{00}$ is given by

$$\tau_n(t) = \frac{(\Gamma(t)U^n(0))_{00}}{(\Gamma(t))_{00}}. \tag{3.17}$$

Then (3.16), (3.17) and (3.11) give a solution to the initial value problem for the Schur flows. Another set of formulae that represents solutions via Hankel determinants and is connected with the Lax representation (2.3) was obtained in [10]. We should also mention papers [6, 7], where an approach based on continuous fractions was used to linearize equations (1.1) in the semi-infinite case.

Note that, since the right-hand side of (3.11) is a homogeneous rational expression in τ_i of degree 0, solutions $\gamma_n(t)$ can be obtained by substituting functions

$$\tilde{\tau}_n(t) = (\Gamma(t)\,U^n(0))_{00} \tag{3.18}$$

into (3.11) instead of $\tau_n(t)$.

In the positive-definite case (2.6) equation (3.18) represent moments of the measure

$$\mathrm{d}\rho(z,t) = \mathrm{e}^{2t\,\mathrm{Re}(z)}\,\mathrm{d}\rho(z,0) \tag{3.19}$$

where $\mathrm{d}\rho(z,0)$ is the measure that corresponds to the initial data $\tau_n(0)$.

Equations (3.14) and (3.19) show that if one 'projects' $\mathrm{d}\rho(z,t)$ into a measure $\mathrm{d}\mu(2\,\mathrm{Re}(z),t) = \mathrm{d}\rho(z,t)$ defined on the real axis, then $\mathrm{d}\mu(\lambda,t) = \mathrm{e}^{\lambda t}\,\mathrm{d}\mu(\lambda,0)$. It is known (see, e.g., [4]), that in this case coefficients of the three-term relations satisfied by monic polynomials orthogonal with respect to $\mathrm{d}\mu(\lambda,t)$ satisfy equations of the Toda lattice (1.2). Explicit expressions for these coefficients in terms of the Schur parameters γ_i can be obtained via the theory of *canonical moments* developed in [9]:

$$\begin{aligned}a_i &= (1-\gamma_{2i})(1-\gamma_{2i+1}^2)(1+\gamma_{2i+2})\\ b_i &= (1-\gamma_{2i-1})(1+\gamma_{2i}) + (1-\gamma_{2i})(1+\gamma_{2i+1}).\end{aligned} \tag{3.20}$$

An alternative way to obtain (3.20) as well as other maps from the Schur flows into the Toda lattice is described in the next section.

4. Schur flows and Toda lattice

First recall the discrete Miura map

$$a_i = \alpha_{2i}\alpha_{2i+1} \qquad b_i = \alpha_{2i-1} + \alpha_{2i} \tag{4.1}$$

that transforms a solution α_i of the Volterra lattice (1.3) into a solution of the Toda lattice. The abstract version of this map was studied in [8] in the context of Poisson–Lie groups. In particular, it was shown that the map (4.1) as well as its generalizations is Poisson with respect to the suitable Poisson structures. In the case we are dealing with, these structures are given by quadratic Poisson brackets

$$\begin{aligned} &\{a_i, a_{i+1}\}_2 = -a_i a_{i+1} && \{b_i, b_{i+1}\}_2 = -a_i \\ &\{a_i, b_i\}_2 = a_i b_i && \{a_i, b_{i+1}\}_2 = -a_i b_{i+1} \end{aligned} \tag{4.2}$$

for the Toda lattice and

$$\{\alpha_i, \alpha_{i+1}\}_l^v = -\alpha_i\alpha_{i+1} \tag{4.3}$$

for the Volterra lattice. As usual, we list only non-zero brackets.

The linear Poisson structure for the Toda lattice is compatible with (4.2) and has the form

$$\{a_i, b_i\}_1 = -\{a_i, b_{i+1}\}_1 = a_i. \tag{4.4}$$

It turns out that the pullback of (4.4) via (4.1) is no longer local as the following lemma shows.

Lemma 4.1. *Poisson brackets*

$$\{\alpha_i, \alpha_j\}_{nl}^v = (-1)^{i+j} \frac{\prod_{i\leqslant 2k+1\leqslant j} \alpha_{2k+1}}{\prod_{i<2k<j} \alpha_{2k}} \qquad (i < j) \tag{4.5}$$

induces, via the Miura map (4.1), the linear Poisson structure for the Toda flow.

Proof. First observe that (4.5) implies

$$\begin{aligned} &\{\alpha_i, \alpha_{2j-1}\}_{nl}^v = -\{\alpha_i, \alpha_{2j}\}_{nl}^v && \text{for} \quad i < 2j-1 \\ &\{\alpha_{2i}, \alpha_j\}_{nl}^v = -\{\alpha_{2i+1}, \alpha_j\}_{nl}^v && \text{for} \quad 2i+1 < j \\ &\alpha_{2j+1}\{\alpha_i, \alpha_{2j}\}_{nl}^v = -\alpha_{2j}\{\alpha_i, \alpha_{2j+1}\}_{nl}^v && \text{for} \quad i < 2j \end{aligned}$$

and

$$\alpha_{2i-1}\{\alpha_{2i}, \alpha_j\}_{nl}^v = -\alpha_{2i}\{\alpha_{2i-1}, \alpha_j\}_{nl}^v \qquad \text{for} \quad 2i < j.$$

These relation ensure that for a_i, b_i defined by (4.1),

$$\{b_i, b_j\}_{nl}^v = \{a_i, a_j\}_{nl}^v = 0 \qquad \text{for any} \quad i, j$$

and

$$\{b_i, a_j\}_{nl}^v = 0 \qquad \text{for} \quad |i - j| > 1.$$

Furthermore,

$$\{a_i, b_i\}_{nl}^v = \alpha_{2i}\{\alpha_{2i+1}, \alpha_{2i}\} = \alpha_{2i}\alpha_{2i+1} = a_i$$

and

$$\{a_i, b_{i+1}\}_{nl}^v = \alpha_{2i}\{\alpha_{2i+1}, \alpha_{2i+2}\} = -\alpha_{2i}\alpha_{2i+1} = -a_i. \qquad \square$$

It follows from lemma 4.1 and compatibility of linear and quadratic Poisson structures for the Toda lattice that (4.3) and (4.5) are also compatible.

Now we can establish a connection between the Schur flows and the Toda lattice using the Volterra lattice as an intermediate step. In fact, we use Darboux-type transformations to describe four ways to construct a solution of the Volterra lattice from the solution of (2.1). Denote

$$\kappa_i^{\pm} = 1 \pm \gamma_i \tag{4.6}$$

and define bidiagonal matrices

$$\begin{aligned} N_1^- &= \left(-\kappa_i^+ \delta_i^j + \delta_i^{j+1}\right)_{i,j=0}^N & \qquad B_1^- &= \left(-\kappa_i^- \delta_i^{j-1} + \delta_i^j\right)_{i,j=0}^N \\ N_1^+ &= \left(\kappa_i^- \delta_i^j + \delta_i^{j+1}\right)_{i,j=0}^N & \qquad B_1^+ &= \left(\pm\kappa_{i-1}^+ \delta_i^{j-1} + \delta_i^j\right)_{i,j=0}^N \end{aligned} \tag{4.7}$$

$$\begin{aligned} N_2^- &= \left((-1)^i(1-(-1)^i\gamma_i)\delta_i^j + \delta_i^{j+1}\right)_{i,j=0}^N \\ B_2^- &= \left((-1)^i(1+(-1)^i\gamma_i)\delta_i^{j-1} + \delta_i^j\right)_{i,j=0}^N \\ N_2^+ &= \left((-1)^i(1+(-1)^i\gamma_{i-1})\delta_i^j + \delta_i^{j+1}\right)_{i,j=0}^N \\ B_2^+ &= \left((-1)^{i-1}(1+(-1)^i\gamma_i)\delta_i^{j-1} + \delta_i^j\right)_{i,j=0}^N. \end{aligned} \tag{4.8}$$

Denote by J a diagonal matrix $\mathrm{diag}((-1)^i)_{i=0}^N$.

Proposition 4.2. *Let L be defined as in (2.2). Then*

$$B_1^+ N_1^+ = L + 2\mathbf{1} \qquad N_1^- B_1^- = L - 2\mathbf{1} \qquad B_2^+ N_2^+ = N_2^- B_2^- = L \tag{4.9}$$

while

$$N_1^+ B_1^+ = A_1^+ + 2\mathbf{1} \qquad B_1^- N_1^- = A_1^- - 2\mathbf{1} \tag{4.10}$$

and

$$N_2^+ B_2^+ = A_2^+ + 2J \qquad B_1^- N_1^- = A_2^- - 2J \tag{4.11}$$

where

$$A_1^{\pm} = \left(\alpha_i^{\pm}\delta_i^{j-1} + \delta_i^{j+1}\right)_{i,j=0}^N \qquad A_2^{\pm} = \left(\tilde{\alpha}_i^{\pm}\delta_i^{j-1} + \delta_i^{j+1}\right)_{i,j=0}^N \tag{4.12}$$

and

$$\begin{aligned} \alpha_i^- &= \kappa_i^- \kappa_{i+1}^+ & \qquad \alpha_i^+ &= \kappa_{i-1}^+ \kappa_i^- \\ \tilde{\alpha}_i^- &= -(1+(-1)^i\gamma_i)(1+(-1)^i\gamma_{i+1}) & \qquad \tilde{\alpha}_i^+ &= -(1+(-1)^i\gamma_i)(1+(-1)^i\gamma_{i-1}). \end{aligned} \tag{4.13}$$

Furthermore, if γ_i satisfy (2.1), then $\alpha_i^{\pm}$ and $\tilde{\alpha}_i^{\pm}$ satisfy equations of the Volterra lattice with boundary conditions $\alpha_0^{\pm} = \alpha_N^{\pm} = 0$ or, equivalently, $A_{\pm}$ satisfy the Lax equation

$$\dot{A}_{1,2}^{\pm} = [A_{1,2}^{\pm}, \pi_+((A_{1,2}^{\pm})^2)]. \tag{4.14}$$

The non-local bracket (2.11) induces the Poisson bracket $\{\ ,\ \}_l^v$ for $\alpha_i^{\pm}$ and the Poisson bracket $\{\ ,\ \}_l^v + 4\{\ ,\ \}_{nl}^v$ for $\tilde{\alpha}_i^{\pm}$.

Proof. Once formulae (4.7) and (4.8) are given, relations (4.9)–(4.14) are not hard to verify.

Let us check relations (4.3) for α_i^- (the case of α_i^+ can be treated similarly). We use notation (2.13). Then (2.11) implies, for $i < j$

$$\{\gamma_i, \kappa_j^+\kappa_{j+1}^-\}_{nl} = \pi_{i+1,j-1}\big(-\kappa_i^-\kappa_j^-\kappa_{j+1}^- + \kappa_j^+\kappa_i^-\kappa_{j+1}^-(\kappa_j^-/\kappa_j^+)\big) = 0.$$

Therefore,

$$\{\alpha_i^-, \alpha_j^-\}_{nl} = 0 \qquad \text{for} \quad i < j - 1$$

and

$$\{\alpha_i^-, \alpha_{i+1}^-\}_{nl} = \kappa_i^+\kappa_{i+1}^+\{\kappa_{i+1}^-, \kappa_{i+2}^-\}_{nl} = -\alpha_i^-\alpha_{i+1}^-.$$

To compute the Poisson algebra induced by (2.11) for $\tilde{\alpha}_i^{\pm}$, let us first consider for $i < k$

$$\begin{aligned}\{\tilde{\alpha}_{2i}^-, \tilde{\alpha}_{2k}^-\}_{nl} = -\pi_{2i+2}^{2k-1}\big(&\kappa_{2i}^-\kappa_{2i+1}^-\kappa_{2k}^-\kappa_{2k+1}^+ + \kappa_{2i}^+\kappa_{2i+1}^-\kappa_{2k}^-\kappa_{2k+1}^- + \kappa_{2i}^+\kappa_{2i+1}^-\kappa_{2k}^-\kappa_{2k+1}^+ \\ &+\kappa_{2i}^-\kappa_{2i+1}^-\kappa_{2k}^-\kappa_{2k+1}^-\big).\end{aligned}$$

Since $\kappa_i^- + \kappa_i^+ = 2$ for any i, the right-hand side of the last equation is equal to

$$-4\kappa_{2i+1}^-\kappa_{2k}^-\pi_{2i+2}^{2k-1} = 4\frac{\tilde{\alpha}_{2i+1}^-\tilde{\alpha}_{2i+3}^-\cdots\tilde{\alpha}_{2k-1}^-}{\tilde{\alpha}_{2i+2}^-\tilde{\alpha}_{2i+4}^-\cdots\tilde{\alpha}_{2k-2}^-} = 4\{\tilde{\alpha}_{2i}^-, \tilde{\alpha}_{2k}^-\}_{nl}^v.$$

Other Poisson brackets for $\tilde{\alpha}_i^{\pm}$ can be computed similarly. □

The Bäcklund transformation $(\gamma_i) \to \alpha_i^{\pm}$ can be found, for example, in [13], whereas the transformation $(\gamma_i) \to \tilde{\alpha}_i^{\pm}$ is, to the best of our knowledge, new. A superposition of any of these maps with the Miura transformation (4.1) transforms a solution of the finite Schur flows into a solution of the finite non-periodic Toda lattice. Moreover, it follows from lemma 4.1 and proposition 4.2 that non-local Poisson structure (2.11) is connected with both linear and quadratic Poisson structures for the Toda lattice.

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